Lecture 5

Wednesday, 11 February 2026 5:56 PM

Duality, Farkas' Lemma, Applications.

Farkas Lemma: Let $A_1, \dots, A_n, b \in \mathbb{R}^m$.
Then exactly one of the following holds:

① $\exists x \in \mathbb{R}^n_+$ s.t. $\sum_{j=1}^n A_j x_j = b$

② $\exists y \in \mathbb{R}^m$ s.t. $\forall j,\ y^T A_j \geq 0,\ b^T A_j < 0$

Geometric perspective: either b lies in the cone formed by A_j's, or b lies outside & $\exists$ a separating hyperplane (w/ normal vector y)

Proof of strong duality using FL:

(**Strong duality**:

$$\min c^T x \qquad\qquad \max b^T y$$
$$Ax \geq b \qquad\qquad A^T y = c$$
$$y \geq 0$$

Let x^* be an optimal primal soln. Then $\exists$ an optimal dual soln y^*, and $c^T x^* = b^T y^*$.)

Proof of strong duality: Let x^* be optimal soln,
$a_1^T x^* = b_1, \dots, a_k^T x^* = b_k$ be all the tight constraints. Note that $\forall d$:
$$a_1^T d \geq 0, \dots, a_k^T d \geq 0,$$
it must be true that $c^T d \geq 0$

By FL, $\exists y_1^* \dots y_k^* \geq 0$ s.t. $c = \sum_{i=1}^k a_i y_i^*$

set $y_{k+1}^* = \dots = y_m^* = 0$

the $c = \sum_{i=1}^m a_i y_i^*$, $y^* \geq 0$
$$= A^T y^*$$

& $c^T x^* = y^{*T} A x^* = \sum_{i=1}^k y_i^* a_i^T x^*$
$$= \sum_{i=1}^k y_i^* b_i = b^T y^* \qquad \blacksquare$$

Proof of FL:

Easy part: ① $\Rightarrow \neg$②:

$\exists x \geq 0 : \sum_{i=1}^n A_i x_i = b$

$\Rightarrow \forall y \in \mathbb{R}^m$,
$$y^T A_i \geq 0, \Rightarrow y^T b = \sum_{i=1}^n y^T A_i x_i \geq 0$$

Will show $\neg$① $\Rightarrow$ ②

i.e., $\nexists x \geq 0 : \sum_{i=1}^n A_i x_i = b \Rightarrow \exists y \in \mathbb{R}^m : \forall i \in [n]$
$$y^T A_i \geq 0,\ y^T b < 0.$$

Let $P = \{ Ax : x \geq 0 \}$, then $b \notin P$

Consider the problem:
$$\min \|b - z\|_2^2 : z \in P$$

Problem is P is not bounded, but we can fix that:
$0 \in P$, hence objective $\leq \|b\|_2^2$

So, $\min \|b - z\|_2^2 : z \in P,\ \|b - z\|_2^2 \leq \|b\|_2^2$

This has a (unique) soln, say z^*.

Then let $y = z^* - b \neq 0$. Will show this is almost reqd. y.

Then $\forall z \in P$, $\lambda z + (1-\lambda) z^* \in P$ for $\lambda \in [0,1]$

hence $\|\lambda z + (1-\lambda) z^* - b\|_2^2 \geq \|z^* - b\|_2^2$

or $\lambda^2 \|z - z^*\|_2^2 + \|z^* - b\|_2^2 + 2\lambda (z - z^*)^T (z^* - b)$
$$\geq \|z^* - b\|_2^2$$

or for $\lambda > 0$, $\lambda \|z - z^*\|_2^2 + 2 (z - z^*)^T (z^* - b) \geq 0$

But then must be that $(z - z^*)^T (z^* - b) \geq 0$
(else we could make λ really small & get a -ve number)

or $y^T z \geq y^T z^*$
$$= y^T (z^* - b) + y^T b$$
$$> y^T b$$

so $\forall z \in P$, $y^T z > y^T b$

HW: Show how to get $\hat{y}$ s.t. $\hat{y}^T b < 0$, $\hat{y}^T z \geq 0$ $\forall z \in P$

Alternative FL: Given $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$,

① Either $\exists x$ s.t. $Ax = b$,

② Or $\exists y$ s.t. $y^T A = 0$, $y^T b > 0$

(exactly one must hold)

An example of duality: Max-Flow Min-Cut theorem.

Max-flow:

Given digraph $G = (V, E)$, each edge has capacity c_e, spl vertices s & t

A flow $f: E \to \mathbb{R}_+$ satisfies $\forall v \neq s, t$:
$$\sum_{e \in \delta^+(v)} f_e = \sum_{e \in \delta^-(v)} f_e,$$

where $\delta^+(v) = \{\text{edges leaving } v\}$
$\delta^-(v) = \{\text{edges entering } v\}$

& $\forall e$, $f_e \leq c_e$

Define $\text{excess}_f(v) = \sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e$

Then $\sum_{v \in V} \text{excess}_f(v) = 0 \Rightarrow \text{excess}_f(s) = -\text{excess}_f(t)$
(since $\text{excess}_f(v) = 0\ \forall v \neq s, t$)

$|f| :=$ value of flow $= \text{excess}_f(s) = \sum_{e \in \delta^+(s)} f_e - \sum_{e \in \delta^-(s)} f_e$

Problem: Find flow of maximum value.

Define a cut $S \subseteq V$, s-t cut is a cut S s.t. $s \in S$, $t \notin S$.

Capacity of cut $= c(S) = \sum_{e \in \delta^+(S)} c_e$

Problem: Find cut of minimum capacity.

Easy to see that for any s-t cut (S) & flow f,
$$|f| \leq c(S)$$

Thm (Max-Flow Min-cut): Max flow = Min cut.

Max flow as an LP:

$$\max \sum_{e \in \delta^+(s)} f_e - \sum_{e \in \delta^-(s)} f_e$$

$\forall v \neq s,t$ $\quad \sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e = 0 \quad \times \lambda$

$\forall e$ $\quad f_e \leq c_e \quad \times \mu$

$\forall e$ $\quad f_e \geq 0$

$$\min \sum_e c_e \mu_e$$

$\forall e = (u,v)$, $u, v \neq s, t$ $\quad \mu_e + \lambda_v - \lambda_u \geq 0$

$\forall e = (u,t)$ $\quad \mu_e - \lambda_u \geq 0$

$\forall e = (t,v)$ $\quad \mu_e + \lambda_v \geq 0$

$\forall e = (s,v)$ $\quad \mu_e + \lambda_v \geq 1$

$\forall e = (u,s)$ $\quad \mu_e - \lambda_u \geq -1$

$\forall e$ $\quad \mu_e \geq 0$

Can write as:

$$\min \sum_e c_e \mu_e$$

$\forall e = (u,v)$ $\quad \mu_e + \lambda_v - \lambda_u \geq 0$

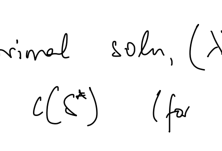
$\lambda_s = 1$

$\lambda_t = 0$

$\mu_e \geq 0$

Let $S \subseteq V$ be an s-t cut.

Let $\lambda_v = 1$ $\forall v \in S$, $\lambda_v = 0$ o.w., $\mu_e = 1$ $\forall e \in \delta^+(S)$



Can check this is dual feasible, & objective $= c(S)$

hence max-flow $\leq$ min-cut.

Will use complementary slackness to show equality.

Now let f^* be optimal primal soln, (λ^*, μ^*) be optimal dual soln. Will show $|f^*| = c(S^*)$ (for some cut S^*)

Let $S^* = \{ v : \lambda_v^* \geq 1 \}$. Then $s \in S^*$, $t \notin S^*$, so this is an s-t cut.

$\forall e \in \delta^+(S^*)$, $e = (u,v)$, $\lambda_u^* \geq 1$, $\lambda_v^* < 1$

so $\mu_e^* \geq \lambda_u - \lambda_v > 0 \Rightarrow f_e^* = c_e$

$\forall e \in \delta^-(S^*)$, $e = (u,v)$, $\lambda_u^* < 1$, $\lambda_v^* \geq 1$

$\mu_e^* \geq 0 \Rightarrow \mu_e^* + \lambda_v^* - \lambda_u^* > 0 \Rightarrow f_e^* = 0$

Hence $c(S^*) = \sum_{e \in \delta^+(S^*)} c_e = \sum_{e \in \delta^+(S^*)} f_e^* - \sum_{e \in \delta^-(S^*)} f_e^*$
$$= \sum_{e \in \delta^+(s)} f_e^* - \sum_{e \in \delta^-(s)} f_e^* = |f^*|$$

Can show that OPT (dual) $= c(S^*)$, completing proof.